Exercise Sets

KS Philosophical Logic: Modality, Conditionals Vagueness

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Exercise Set 1

Propositional and Predicate Logic

Please answer all of the following questions.

1. Use Definition 1.1 (*Handout I Propositional Logic*) to decide whether the following are well-formed formulae. Explain your answers.

   (a) $((p \supset q) \supset (\neg q \supset \neg p))$
   (b) $p \wedge \neg p \supset p$
   (c) $A \supset (B \supset A)$
   (d) $\bot \supset q$

2. Fill in the quotes where necessary to make the following sentences true:

   (a) Graz is what Graz refers to.
   (b) Consist of five words consists of several words.
   (c) There are seven words in this sentence.
   (d) Graz refers to Graz is a sentence about what Graz means.

3. Check the truth of each of the following, using tableaux. If the inference is invalid, read off a counter-model from the tree, and check directly that it makes the premises true and the conclusion false:

   (a) $p \supset q, r \supset q \vdash_{\mathsf{C}} (p \vee r) \supset q$
   (c) $\vdash_{\mathsf{C}} ((p \supset q) \supset q) \supset q$

4. (a) Explain informally why $\forall x Px \vDash_{\mathsf{PL}} \exists x Px$ (cf. Definition 2.1 on *Handout II Predicate Logic*). What would have to be changed in Definition 2.1 of models for PL if we wanted $\forall x A \nvDash_{\mathsf{PL}} \exists x A$? What problems may this change have? (Hint: Check Priest's discussion in §12.6.)
   (b) Explain informally, by appeal to the model theory of PL, why
   $\forall x Px \vee \forall x Qx \vDash_{\mathsf{PL}} \forall x(Px \vee Qx)$, but
   $\forall x(Px \vee Qx) \nvDash_{\mathsf{PL}} \forall x Px \vee \forall x Qx$.

5. Check the truth of the following, using tableaux. If the inference is invalid, use an open branch to specify a counter-model for the inference: $\vdash_{\mathsf{PL}} \forall x Px \equiv \neg \exists x \neg Px$

6. Formalise the following reasoning in first-order logic. Using tableaux, check if the inference is valid. If the inference is invalid, use an open branch to specify a counter-model for the inference. (Use the letters P for 'Catholic', Q for 'Christian', and S for 'creationist'.)

   All Catholics are Christians. Some Christians are creationists. So all Catholics are creationists.

Exercise Set 2

Please answer all of the following questions.

1. Show that the truth value of $\neg\Box A$ at a world is the same as that of $\Diamond\neg A$. (Hint: Use the clauses for $\Box$, $\Diamond$, and $\neg$ of the definition of a valuation for a model of propositional modal logic on Handout *III-1: Propositional Modal Logic*.)

2. Call a world **blind** if it sees no worlds. If a world w is blind, what type of formula is vacuously true? Which is vacuously false?

3. Consider again the definition of validity in system K (Definition 3.4 on *Handout III-1*):

   We say that a world w of model $\mathcal{M}(=\langle\mathcal{W},\mathcal{R},\mathcal{J}\rangle)$ **models** formula A just in case the given formula is true at that world on that model, i.e. $\nu_{\mathcal{M},w}(A)=1$.

   Let $\mathcal{M}$ be a model $\langle\mathcal{W},\mathcal{R},\mathcal{J}\rangle$. We say that a formulae A is **true in** $\mathcal{M}$ iff for every world $w\in\mathcal{W}$, $\nu_{\mathcal{M},w}(A)=1$.

   Using K (for Kripke) to refer to our basic modal logic, we say that an inference is **valid in system K** iff every world of every model that models the premises also models the conclusion; i.e.

   $$\Sigma\vDash_{\mathsf{K}} A \text{ iff for all worlds } w\in W \text{ of all models } \langle\mathcal{W},\mathcal{R},\mathcal{J}\rangle\text{:}$$
   $$\text{if } \nu_{\mathcal{M},w}(B)=1 \text{ for all the premises } B\in\Sigma,\ \text{then } \nu_{\mathcal{M},w}(A)=1$$

   Exercise: Rewrite the definition of validity in system K ('an inference is valid in system K iff ...') by using the notion of *truth in model* $\mathcal{M}$ (as defined) instead of the notion of a world *modeling* a formula on the right-hand side of the biconditional. (Rewrite it in such a way that it is equivalent to the definition as stated above.)

4. The formula $\Box p\supset\Diamond p$ is not valid in system K (i.e. $\nvDash_{\mathsf{K}}\Box p\supset\Diamond p$).

   (a) Find a model $\mathcal{M}(=\langle\mathcal{W},\mathcal{R},\mathcal{J}\rangle)$ that invalidates $\Box p\supset\Diamond p$ (i.e. a counter-model to $\vDash_{\mathsf{K}}\Box p\supset\Diamond p$). Draw a diagram of the model (cf. Priest 2008, §§2.3 and 2.4.8). (Hint: Check §4.1(iv) of *Handout III-1* for a relevantly similar example.)
   (b) Does this fact about K make it a suitable logic for necessity? Why or why not? (Answer in no more than 200 words.)

5. Test the following, using tableaux. Where the tableau does not close, use it to define a counter-model, and draw this, as in Priest (2008, §2.4.8).

   (a) $\vdash_{\mathsf{K}}(\Box p\wedge\Box q)\supset\Box(p\wedge q)$
   (b) $\vdash_{\mathsf{K}}\Diamond(p\wedge q)\supset(\Diamond p\wedge\Diamond q)$
   (c) $\Box p,\Box\neg q\vdash_{\mathsf{K}}\Box(p\supset q)$
   (d) $\Diamond p,\Diamond q\vdash_{\mathsf{K}}\Diamond(p\wedge q)$

Exercise Set 3

Propositional Modal Logic

Please answer <u>all</u> of the following questions.

1. Consider normal systems of propositional modal logic K, D, T, B, $\mathsf{S4}$, $\mathsf{S5}$. Remember that a model for any normal propositional modal logic is a structure $\langle \mathcal{W}, \mathcal{R}, \mathcal{J} \rangle$ (cf. Def. 3.1 on *Handout III-1*).

   (a) Find a T-model in which '$\Box p \supset \Box\Box p$' is false.
   (b) Find an B-model in which '$\Diamond p \supset \Box\Diamond p$' is false.
   (c) Find an $\mathsf{S5}$-model in which '$\Diamond p \supset \Box p$' is false.

2. What is the weakest modal logic system in which the following formulae are theorems? (Hint: Test using tableaux and check which rules additional to those of K you needed.)

   (a) $\vdash_{?}\ \Diamond p \supset \Diamond\Diamond p$
   (b) $\vdash_{?}\ (\Box p \wedge \Box q) \supset (p \equiv q)$

3. $\mathcal{R}$ is reflexive (ρ), it is serial (η). Hence, if truth is preserved at all worlds of all D-models (= serial models), it is preserved at all worlds of all T-models (= reflexive models). Consequently, the system T is an extension of the system D. Find an inference (from at least one premise) demonstrating that system T is a *proper* extension of D. (That is, find an inference and show, using tableaux, that it is a proof in T but not in D.)

4. Test the following inferences using tableaux. If a tree does not close, use an open branch to define a counter-model. (Note the subscripts CK/VK on '$\vdash$'.)

   (a) $\Box\Diamond\exists x Px \vdash_{\mathsf{CK}} \Box\exists x\Diamond(Px \vee Qx)$
   (b) $\vdash_{\mathsf{VK}}\ \Diamond\forall x Px \supset \forall x\Diamond Px$

5. Consider the following inference from *Handout IV-1*: $\forall x\Box(Px \supset Qx) \nvdash_{\mathsf{CK}} \forall x(Px \supset \Box Qx)$

   What happens if we add the ρ constraint (cf. *Handout III-2*, §2.2)? Test this using a tree with the ρ-rule. Does this have any impact on the result? Is the inference a proof in this system (i.e. in quantified modal logic CK_ρ? If the tree is open, read off a counter-model from an open branch.

6. Consider an instance of the Converse Barcan Formula (CBF): $\Box\forall x Px \supset \forall x\Box Px$

   (a) Is CBF an intuitively plausible principle that we want to be a logical truth of QML? Why or why not? (State your answer in no more than 200 words. It might be a helpful to use an example.)
   (b) Is CBF a logical truth (valid) of constant domain quantified modal logic CK? Is CBF a logical truth (valid) of variable domain quantified modal logic VK? (You do *not* need to explain your yes/no answers.)

Exercise Set 4

Conditionals: Material & Strict; Grice

Please answer all of the following questions.

1. (a) Give two examples of your own of conditionals *in German* that do not contain the word ‘wenn’. (If you’re not a native speaker of German, give your own examples of conditionals without ‘if’ in English.)

   (b) Give your own example of a pair of conditionals in English or German . . .
   • which differ only in that one is in indicative and the other in subjunctive mood, and
   • one of which is intuitively true while the other is intuitively false.
   (See example (8a/b) on *Handout V-1* for relevant illustration.)

2. Give your own (English or German) example of the following inference pattern that shows its intuitive invalidity:

$$(A \wedge B) \rightarrow C \;\; \Rightarrow \;\; (A \rightarrow C) \vee (B \rightarrow C)$$

3. Check, by using tableaux, whether the following inference pattern is invalid in normal modal logics stronger than K. In your answer, state explicitly which system is the strongest modal logic in which the inference pattern is invalid.

   (Hint: (i) Replace any formula ‘$A \prec B$’ with ‘$\Box(A \supset B)$’ on the tree. (ii) Go from stronger to weaker logics: If an inference pattern is invalid in a stronger system, it is invalid in a weaker system.)

$$\neg(A \prec B) \nvDash_{\mathsf{K}} A$$

4. Consider the quote from C.I. Lewis (cf. *Handout V-1*):

   > “‘Proof’ requires that a connection of content or meaning or logical connection be established. And this is not done for the postulates and theorems in material implication . . . For a relation which does not indicate relevance of content is merely a connection of ‘truth-values’, not what we mean by a ‘logical’ relation or ‘inference’.” (Lewis, 1917, 355)

   Does Lewis’ own proposal for the meaning of *if . . . (then)* – i.e., strict implication – succeed in establishing a relation that ‘indicate[s] relevance of content’ (of antecedent and consequent)? Why or why not? Give an example in English or German to support your answer. Answer in no more than 200 words.

5. Give your own example, in English or German, of an assertion that under normal circumstances carries a *conversational* implicature. State (i) the sentence asserted, (ii) what, according to Grice, it *says* (its conventional/literal/semantic meaning), and (iii) what it conversationally implicates.

6. Grice (1989, 58-9) maintains that the Indirectness Condition is non-detachable, and he gives the following examples to support his claim:

   (1) Either Smith is not in London, or he is attending the meeting.
   (2) It is not the case that Smith is both in London and not attending the meeting.

   According to Grice, (1) and (2) – both of which *say* the same as 'If Smith is in London, he's attending the meeting' (they're truth-functionally equivalent to 'Smith is in London $\supset$ Smith is attending the meeting') – also implicate the Indirectness Condition.

   Give a counter-example to the claim that the Indirectness Condition is a *non-detachable* conversational implicature of natural-language conditionals. That is, give an example in which it is plausible to claim that an assertion of 'if. . . (then)' conversationally implicates the Indirectness Condition but in which truth-functionally equivalent statements clearly fail to carry this implicature.

7. Consider Dorothy Edgington's criticism of Grice's defense of the Supplemented Equivalence Thesis:

   > But the difficulties with the truth-functional conditional cannot be explained away in terms of what is an inappropriate conversational remark. They arise at the level of belief. Believing that John is in the bar does not make it logically impermissible to disbelieve "if he's not in the bar he's in the library". Believing you won't eat them, I may without irrationality disbelieve "if you eat them you will die". Believing that the Queen is not at home, I may without irrationality reject the claim that if she's home, she will be worried about my whereabouts. As facts about the norms to which people defer, these claims can be tested. But, to reiterate, the main point is not the empirical one. We need to be able to discriminate believable from unbelievable conditionals whose antecedent we think false. The truth-functional account does not allow us to do this. (Edgington, 1995, 245)

   Is Edgington's criticism a forceful objection against the Gricean account? Give reasons in support of your answer. Answer in no more than 250 words.

Exercise Set 5

Stalnaker on Conditionals

Please answer all of the following questions.

1. Stalnaker (1975, 63) writes:

> Or consider what may be inferred from the *denial* of a conditional. Surely I may deny that if the butler didn't do it, the gardener did without affirming the butler's innocence. Yet if the conditional is material, its negation entails the truth of its antecedent.

Write down the inference schema, using formal notation ('$\rightarrow$ for the conditional, $\Rightarrow$ for validity), of which Stalnaker gives the above example and claims that the conclusion doesn't intuitively follow from the premise. (Hint: We have already come across this inference pattern.)

2. (a) Show that in Stalnaker's logic C_2, the following inference is valid. Since there is presently no known tableaux system for C_2,[1] you need to show this by *reasoning semantically*: take any way to make the premises true and show that it also makes the conclusion true. (Hint: For examples of semantic reasoning of this kind, see *Handout VI*, §5.)

   $A \wedge B \vDash_{\mathsf{C}_2} A > B$

   (b) Is the following inference valid in C_2? If it is valid, show that it is by reasoning semantically (see above). If it is invalid, show that it is by constructing a counter-model directly, by trial and error — try to make the premise true and the conclusion false at a world in the model. (Hint: check *Handout VI*, §5 for relevant examples. The degree of formal rigor in the presentation of the counter-model on the Handout is sufficient for your answer.)

   $A > B \vDash_{\mathsf{C}_2} A \supset B$

   (c) Is *Modus Ponens* valid in C_2? Show whether it is valid or invalid by reasoning semantically.

   Modus Ponens: $A, A > B \vDash_{\mathsf{C}_2} B$

   (d) Is *Modus Tollens* valid in C_2? Show whether it is valid or invalid by reasoning semantically.

   Modus Tollens: $A > B, \neg B \vDash_{\mathsf{C}_2} \neg A$

3. Stalnaker says about his contextual condition (5) on the selection function:

> 'The idea is that when a speaker says "If A," then everything he is presupposing to hold in the actual situation is presupposed to hold in the hypothetical situation in which A is true. Suppose it is an open question whether the butler did

[1] Cf. Priest (2008, 93)

> it or not, but it is established and accepted that whoever did it, he or she did it with an ice pick. Then it may be taken as accepted and established that if the butler did it, he did it with an ice pick.' (Stalnaker, 1975, 69)

Can you think of instances – parallel to the butler example in the quote – where condition (5) leads to conditionals being accepted and established in context but which, intuitively, should *not* be accepted?

4. Stalnaker (1975) gives the same semantic analysis of indicative and subjunctive conditionals. How does Stalnaker explain the difference between between indicative and subjunctive conditionals? (Answer in no more than 250 words.)

5. In Stalnaker's logic C_2, the **Limit Assumption** holds:

   Limit Assumption: For every possible world w and every nonempty proposition A, there is at least one A-world *most* similar to w.

   David Lewis objects to the Limit Assumption as follows:

   > 'Unfortunately we have no right to assume that there always are a smallest antecedent-permitting sphere and, within it, a set of closest antecedent worlds. Suppose we entertain the counterfactual supposition that at this point
   >
   > there appears a line more than an inch long. (Actually it is just under an inch.) There are worlds with a line $2''$ long; worlds presumably closer to ours wit ha line $1\frac{1}{2}''$ long; worlds presumably still closer to ours with a line $1\frac{1}{4}''$ long; worlds presumably still closer ... But how long is the line in the *closest* worlds with a line more than an inch long? If it is $1+x''$ for any x however small, why are there not other worlds still closer to ours in which it is $1 + \frac{1}{2}x''$, a length still closer to its actual length? The shorter we make the line (above $1''$), the closer we come to the actual length; so the closer we come, presumably, to our actual world. Just as there is no shortest possible length above $1''$, so there is no closest world to ours among the worlds with lines more than an inch long, and no smallest sphere permitting the supposition that there is a line more than an inch long.' (Lewis, 1973, 20-1)

   (a) Give *your own* example that supports Lewis' claim that 'we have no right to assume that there always are [...] a set of closest antecedent worlds.' (Answer in no more than 100 words.)

   (b) Evaluate Lewis' objection. Do you think Lewis' criticism of the Limit Assumption is correct? Give reasons for your answer. (Answer in no more than 200 words.)

Exercise Set 6

Vagueness: The Sorites Paradox & Many-Valued Logic

Please answer all of the following questions.

1. (a) Give four (4) examples of your own of vague expressions in English or German: two adjectives and two nouns.
   (b) Give two (2) examples of *adjectives* in English or German that are *not* vague.
   (c) Construct a Sorites argument from one of the expressions chosen in (1a).

2. Observe that in the logic K_3 if an interpretation assigns the value i to every propositional letter that occurs in a formula, then it assigns the value i to the formula itself.
   (a) Show from this fact that there are no logical truths in K_3.
   (b) Are there any logical truths in Ł$_3$? If so, name one.

3. Describe one important difference between K_3 and Ł$_3$. Given this difference, which logic do you think is the better one, and why? (Answer in no more than 200 words.)

4. Consider Monotonicity:

   Monotonicity: If x is F and x' is F-er than x, then x' is F.

   An instance of Monotonicity is: 'If Susan is tall and Taylor is taller than Susan, then Taylor is tall.'

   Show whether Monotonicity is a valid principle
   (a) in K_3
   (b) in Ł$_3$.

5. Is there a problem for multi-valued/fuzzy logics that is analogous to the problem with higher-order vagueness that besets three-valued logics? Answer in no more than 200 words.

6. Explain how a multivalued/fuzzy logician rejects the Sorites argument as invalid. That is, show what is wrong with the Sorites paradox according to multivalued/fuzzy logic.

7. In a supervaluationist logic, the Law of Excluded Middle (LEM) is valid.
   Show that

   $$\alpha \text{ is either a heap or } \alpha \text{ is not a heap}$$

   is TRUE (i.e. true on all sharpenings).

References

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